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THE MATHEMATICS TEACHER

A MAGAZINE DEVOTED TO THE
INTERESTS OF TEACHERS OF MATHEMATICS

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THE MATHEMATICS TEACHER

A MAGAZINE DEVOTED TO THE
INTERESTS OF TEACHERS OF MATHEMATICS

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EDITED BY
W. H. METZLER

ASSOCIATED WITH
EUGENE R. SMITH JONATHAN T. RORER

VOLUME I MARCH, 1909 NUMBER 3

THE AIMS OF TEACHING ALGEBRA AND HOW TO ATTAIN THEM.

BY C. E. BIKLÉ.

(Concluded.)

The work of many candidates is vitiated by the use of the continued inequation, as

$$x^8 - y^8 = (x^4 + y^4)(x^4 - y^4) = (x^2 + y^2)(x^2 - y^2) = (x + y)(x - y).$$

This kind of error is in part due to the careless work tolerated in arithmetic, as

$$2 \times 3 = 6 + 4 = 10 \div 5 = 2, \text{ and } 6\tfrac{4}{8} = \tfrac{1}{2}.$$

Adding and subtracting fractions are confused with clearing an equation of fractions.

Errors in numerical work abound, and many who are able to give the quadratic formula, and the general term in $(a + b)^n$ are unable to evaluate accurately.

I shall never forget the answers given by many candidates in 1902 to the questions "What do you understand by a^0? by a^{-1}? by $a^{\frac{1}{2}}$?" With confidence came the answers. "I understand that a^0 means a taken zero times as a factor, a^{-1} means a taken minus once as a factor, and $a^{\frac{1}{2}}$ means a taken one half a time as a factor." These pupils had evidently been well-drilled upon the definitions in the first part of the book, to the neglect, perhaps, of training in clear thinking. The problems leading to equations are still avoided.

Means and Remedies.

A few days ago I received a postal card, informing me that a prominent western principal had had no failures in algebra in his school since he had adopted a certain text-book, although he had formerly had a large percentage of failures in that subject. This text-book, like other recent ones, has good points, but the presence of pictures and so-called real problems in the text-book will not relieve the teacher of the necessity of getting the pupils to think.

A live teacher who knows his subject and enjoys teaching it, and who is in the habit of consulting such helps as Chrystal, Oliver Wait and Jones, Fine, Smith and Young will be successful. His text-book may well contain problems taken from many books, directions for correlation of topics, and historical notes. A card-index of problems will be of use to him.

Just before I commenced teaching algebra it was my good fortune to take some work in mathematics with Dr. Durell. One reference book that I used was Oliver Wait and Jones. I believe that the ideas I got from the early pages of that book and from the "Drill-book in Algebra" by Professor Jones have been a great help to me.

To quote from the book first mentioned: "The symbols explained below constitute a symbolic language, a species of shorthand writing wherein numbers and their relations are more conveniently expressed than in the language of words. The signs stand for words and phrases, and generally have the same grammatical relations as the words and phrases themselves. The words may be restored at any time. The reader should constantly practice translating from one to the other till both are familiar. Among the many advantages of this symbolic langnage are clearness, brevity and generality of statement, the ability to mass directly under the eye and thus to bring before the mind all the steps in a long and intricate investigation, and the facility of tracing a number through all the changes it may undergo."

"Read in words the statements below, and point out verbs, conjunctions, etc.

$$(a+b)+(a-b)=2a$$

$$(a+b)-(a-b)=2b$$

$$(x+y)^2-(x-y)^2=4xy$$

$$\frac{a+b}{2}+\frac{a-b}{2}=a$$

$$\frac{a+b}{2}-\frac{a-b}{2}=b$$

$$L-S=D$$

$$L-D=S$$

$$S+D=L.''$$

Emphasizing the sign of operation will help the pupil to see that $ba+ca=(b+c)a$,

$$s_1c+s_2c=(s_1+s_2)c$$

$$\frac{Bh}{2}+\frac{bh}{2}=(B+b)\frac{h}{2}$$

$$\pi R^2-\pi r^2=(R^2-r^2)\pi$$

$$\frac{\pi}{6}D^3-\frac{\pi}{6}d^3=(D^3-d^3)\frac{\pi}{6}$$

Very legitimate work to secure interest in a beginning class is the following problem, taken from Ball's "Mathematical Recreations."

Think of a number.

Multiply it by 5.

Add 6.

Multiply by 4.

Add 9.

Multiply by 5.

Tell me the result, and I will tell you what number you had in mind at first.

The pupil wonders how the teacher is able to tell the number thought of, and a beginning of respect for algebra is secured when he sees that n, the number, is easily found from the result of the last operation, $100n+165$. $\frac{2\pi R+12}{2\pi}-R$ will help the pupil who is trying to find out whether he could slip the blade of his pen-knife between the circumference of the earth and a concentric circumference one foot longer.

The geometric pictures of $(a+b)^2$, $(a-b)^2$, $(a+b)(a-b)$, and such numerical applications as $(121)^2$, $(98)^2$, 125×115, will interest pupils and enlarge their grasp of numbers. I have found that pupils like to build a table of squares by using $(n+1)^2 = n^2 + 2n + 1$.

They are interested in applying factoring immediately to operations with fractions, and in solving equations of the second and higher degrees by using the factor theorem.

They enjoy reviewing their arithmetic problems from the algebraic point of view. The following illustration will serve. Given the times required by men separately to do a piece of work, find the time required when they work together. In the case of two men, with times t_1 and t_2, the result $\frac{t_1 t_2}{t_1 + t_2}$ may give the impression that the product of the times divided by the sum of the times is a formula including the cases in which more than two men are concerned. The result for three men, $\frac{t_1 t_2 t_3}{t_1 t_2 + t_1 t_3 + t_2 t_3}$, removes this impression, and careful study of the equations from which the formula has been obtained brings an interpretation that includes the case in which any number of men is concerned.

By skilful questioning the teacher can help the pupils to think clearly, no matter what type of problems is under consideration. The pupil can usually find the equation which is hidden in the words of the problem if he sets about the work methodically. Very little illustrative material is necessary. The mystery of running water, of moving trains, of the courier riding to the head of the army, may be made clear without bringing into the class-room miniature rivers, railways or soldiers. The pupils must be taught to question themselves and to use diagrams as helps. The questioning is not unlike that used in geometry, the aim in each case being progress through the use of equations.

Pupils know that the time required to go in an automobile plus the time required to walk back will give the number of hours that have elapsed since the start, and most of them know the difference between rowing down stream and up stream.

While the pupils must work and check many problems having no content in themselves in order to acquire facility in opera-

ting with symbols, teachers may prevent the work from becoming mechanical by requiring the pupil to state the reason for each step. This takes time, but it increases the pupil's power of expression, and leads him to see that algebra is a science, and not a mere juggling with symbols.

Right here, it seems to me, is the secret of successful work in algebra. Teach algebra as a science, and classes will take greater interest in the subject and give greater returns in ground covered. Emphasize such points as the number of roots to be expected from an equation; show that $a^0 = 1$ is not a mystery, but a most reasonable convention.

Teach devices, and urge the pupils to use methods that require thought rather than manual labor. Keep in mind the mottoes. "Take a good look at the problem," and "Don't multiply until you have to," using such problems as

$$\frac{(a+b)^2(x+1)+(a+b)(x+1)+(x+1)}{a+b+1} = (a+b)^2+(a+b)+1$$

$$\frac{x+3}{x+1} - \frac{x+4}{x+2} = \frac{x-5}{x-3} - \frac{x-6}{x-4}$$

$$18(x+1)^2(x+2)^2 = 8(x-3)^2(x+1)^2$$

Above all, let the teacher himself know and love his subject. In many classes the work drags because the teacher is merely serving time. Let him know more than he is called upon to teach, and let him give his pupils some idea of the part which algebra is to play in their future work.

CENTRAL HIGH SCHOOL,
SYRACUSE, N. Y.

MODERN TENDENCIES IN THE TEACHING OF ALGEBRA.*

BY FLETCHER DURELL.

At present two movements are prominent, or at least noticeable, in the teaching of algebra: (1) The movement which would make the content or subject-matter less artificial and

* Read at the meeting of the New York Section.

more practical, or which in other words would put algebra in closer accord with the practical sciences and the breadwinning trades (closely connected with this and perhaps part of it are the correlation and laboratory movements); (2) the movement which looks to a better logical development of algebra, a closer definition of its concepts, and a more rigorous treatment of its principles.

But there is another aspect of the matter which is bound strongly to influence the movements just named, even if in time it does not come to underlie and include them. This may be called the utility or functional method of treating the teaching of algebra. This method puts the uses, offices or functions of algebraic concepts and processes in the first place, and the exactness of their relations, that is, proofs of all kinds, in the second. According to it, an axiom is not so much a fundamental equivalence or source of certainty as it is a fundamental economy or tool. Even when an axiom is a source of certainty, this certainty is but the means to new utility. The functional method is largely coincident with what is called the humanistic method which is attracting attention in education. The humanistic method is so called because it appeals not merely to narrow intellectualism on the one hand or to narrow industrialism on the other, but to the whole man. The functional or humanistic method puts the physiology of a subject before its anatomy, uses and functions before structures. It is closely related to the philosophy of pragmatism which is itself an outgrowth or consequence of the philosophy of evolution. It is influencing other departments of education, as may be noted in such text-books as Millikan and Gale's "Physics," Reed and Kellogg's "English Grammars," and Judd's "Psychology," and seems likely to have a profound influence on education as a whole.

We will now give a few illustrations with a view to showing what the functional method means when applied to the study of algebra. Our illustrations are selected mainly with a view to making clear our leading idea and not with a view to completeness or consecutive order.

Take, for instance, the case of the zero exponent. In the cremations of subjects of study which freshmen and sophomore classes used to hold twenty years ago, algebra was prominent, and prominent in the cremation of algebra was that of x^0.

Pupils used frequently to ask the question how x^0 could have any meaning; if the exponent denotes the number of times a factor is taken how can a factor be taken zero times? Let us look at this matter from the functional point of view. If x^3 be divided by x^3 from one point of view we get unity as the quotient, or if we subtract exponents we get x^0. It is merely a matter of convenience which form of the result we use. x^0 is not a metaphysical mystery; it is only a convenient and efficient way of writing the result of the division of some power of x by itself. It finds a use, for instance, when we are writing a polynomial expression, in showing that the absolute term may be taken as continuous with the x terms. In other words x^0 is not to be interpreted metaphysically but functionally and practically. After such an interpretation it is accepted as readily and used as freely by pupils as any other power of x.

Similarly a negative exponent is seen to be merely a means or instrument to enable us to write a fraction in a form often more convenient than the customary one; also a fractional exponent may be regarded at the outset as only a convenient way of combining a power and a root of the same number. After preliminary difficulties have been cleared away in this manner, and after pupils have got their general utility bearings, they are much more ready to study the so-called proofs, or what I would call methods of establishing equivalence relations among various kinds of exponents (and hence of setting up indirect routes of applications for their utilities), and also for various later generalizations of exponents, and for a closer logical examination of their nature.

Taking up another illustration—in studying elementary algebra many pupils have difficulty in learning to use a parenthesis and its contents as a single quantity, as for instance in the multiplication of $(2x + 3a) + 5b$ by $(2x + 3a) - 5b$. If we take the functional view of this matter and realize that the office of the symbol ( ) is to combine a number of objects into a whole or unity we realize that the symbol in the present case is not sufficiently expressive of unity to the pupil to accomplish its purpose. Hence we may vary the symbol so as to make it more expressive of its function, convert it into a closed oval and get for example $\overline{(2x + 3a) + 5b}$ by $\overline{(2x + 3a) - 5b}$. If this change still does not make the parenthesis sufficiently expressive,

for it and its contents we may use p, the first letter of the word parenthesis, and write $p + 5b$, $p - 5b$, reversing the substitution after the multiplication.

Similarly the meaning and spirit (function) of other symbols may be grasped, and their use varied and extended. This not only helps in the teaching of algebra but enlarges the field of its application to life.

Let me give a simple illustration of this use of the broader spirit of algebra which may be inculcated. A certain business house had an address of, say, 17–31 Barclay St. They advertised extensively in magazines and wished to be able to compare the returns from the different magazines. They found it impossible to induce any considerable number of their correspondents to state which particular advertisement had attracted their attention. Hence the firm followed the plan of giving a slightly different address for their business in each advertisement. Thus in *Harper's Magazine* they advertised the address of the house as 17 Barclay St., in *McClure's Magazine* as 19 Barclay St., in *Scribner's* as 21, etc. Hence the replies could be grouped by the addresses used on them. This was in effect an automatic algebraic device for grouping objects.

In like manner in the more strictly cultural field this method if fully carried out leads to a realization of all language (except the fine arts in part) as essentially a form of algebra, and the study of algebra comes to throw a new and fundamental light on the study and use of language.

As another illustration let us now apply our principle of putting the offices or functions of objects in the first place to the teaching of imaginaries. We first clear away the old metaphysical difficulties that hang around the topic by asking the pupil to set aside the psychological-metaphysical word "imaginaries" and substitute for it a visual graphical term like "perpendicular numbers." Thus we explain in the usual way that if OA is $+1$ and OA' is -1, we may conceive OA as converted into OA' by a rotation through 180°. But we may also conceive this rotation as subdivided into two steps or two rotations of 90° each, one of which, if performed on the result of the first, converts OA into OA'. Hence, since $\sqrt{-1}$ means a rotation through 90°, when applied once to OA it gives OB in the perpendicular position. But from the broad functional point of

view it means more than this: it means that instead of dealing with simple positive and negative numbers represented by a line running to the right and left, we have enlarged our domain of number to something infinitely greater—a plane surface. Hence in the use of imaginaries we have found a vast storehouse of auxiliary quantities to draw on. If the annexation of negative numbers to the positive numbers of arithmetic has already given us so much new power in early algebra, what ought not this infinitely larger magazine of auxiliary quantity to do for us? After pupils have thus got their broad utility bearings they are interested in the means used to develop the power latent in this storehouse of toolage. They readily accept the mechanical rule that as compared with radicals the only change in methods of operation is to make the product of the two minus signs under a radical sign a minus sign outside (for example $\sqrt{-2} \times \sqrt{-3} = -\sqrt{6}$); accept the use of i as an abbreviation, and the use of the cycle or group of powers of i, etc. The pupil by this time has become indifferent to the name by which these quantities are known. You may call a concept or tool what you please provided the pupil understands its functions or uses. Thus in the treatment of this topic the use of this idea of functionality tends to remove some difficulties, to arouse a broad and pervasive interest, to give large cultural values, and at the same time to lead up to technical applications for special classes of students.

As another illustration let us consider next a formula like the one for the sum of n terms of an A.P., $S = n/2(a + l)$. The direct function of this formula is to convert addition into multiplication, a longer process into a shorter. Even so obvious a utility as this seems seldom to be brought specifically to the pupil's attention, it being left to filter in as a result of mere impact. But we also have here an example of a deeper functionality and one whose importance can hardly be overestimated. Why is it that if we know the two formulas $l = a + (n - 1)d$, $S = n/2(a + l)$, we know all of A.P.? Why is it that the three cases of percentage can be combined into one by formula-equation $p = b \times r$, and the four cases of interest into another, viz: $p = i \cdot r \cdot t$. This result is obtained not merely because the symbols involved are small and the formula a kind of shorthand. An important part of the explanation lies in the fact that the symbols p, b, r are approximately uniform in size and

that there is among them not only uniformity of size, but symmetry in their positional relations. Hence the eye not only grasps them all simultaneously but is ready to treat all alike, giving each the lead in turn, etc. A realization of this enables the pupil to understand that in each formula of science he not only has a shorthand expression for a multiplitude of phenomena, but that he also has many formulas in one, as many as there are letters in the given formula. It becomes plain in time also to him that these useful properties or functions give rise to a dialectic or aggressive onward development of other functions, as by the generalization of coefficients or exponents or number of terms, by the combination of more or less similar formulas, and by the re-use of their properties or of the properties of the phenomena which one represents in connection with another. When the many useful functions of formula-equations are thus realized, pupils not only have no dislike of them, but realize them as a useful help and are glad to state a group of related phenomena in the formula shape whenever they can.

Let us now apply the idea of functionality to the fundamental laws of algebra, called the associative, distributive and communative laws. Here again we may begin by clearing away all ideas of metaphysical necessity in these laws. This may be done by the use of the conventional illustrations or by purely functional illustrations such as the following which seem to appeal more strongly to the pupil. Thus we may bring out that so far as actual work is concerned the multiplication of 3 by 7,295 does not equal 7,295 multiplied by 3. That is, the multiplication of 7,295 by 3 is equivalent to the multiplication of 3 by 7,295 in some respects but not in others. Also in using a table (a traverse table, for instance) where the last figure is approximate sometimes $10 \times 2 \neq 2 \times 10$.

Thus it is brought home to the pupil that instead of laws these principles might perhaps better be named customary or conventional methods of arranging symbols. They are used because they save labor and give new power. Thus in transforming $3(a+b)-2(a-b)$ into the successive form $3a+3b-2a+2b$, $3a-2a+3b+2b$, $a+5b$ we are following methods of ordering and combining objects for which we are at liberty to substitute better ones if we can find them.

When grasped in this way it becomes clear that the laws of algebra contain implicit in them important first principles of order, arrangement and organization, and that the study of this part of algebra may have many useful applications to life at large.

Taking up a matter of somewhat different kind, it may be shown that the functional method reduces to system and control, and makes valuable in new ways processes which are usually treated in a more or less haphazard manner and regarded as a matter of common sense.

One of the illustrations of the practical value of geometry which we are accustomed to give is that of determining the height of a tall building or steeple or tree by comparing the length of its shadow with the length of the shadow of some other object whose height is known. Yet how many of us have had occasion to measure the height of any object thus, or have ever earned a dollar or a loaf of bread thus or ever expect to? Nevertheless, there is inherent in this illustration a principle of utility which is of far-reaching importance, viz.: the use of auxiliary quantity to achieve a result difficult or impossible. This principle of the use of adapted auxiliary quantity the physicist uses constantly; so does the chemist. It is omnipresent in all mechanic arts and in business; it is the basis of ethics and philosophy. In fact, we all need to use it every hour, often many times an hour. Anything that will enable us to improve our use of adapted auxiliary quantities or objects is of fundamental and comprehensive importance. Perhaps mathematics, by the relative simplicity of its concepts and by its freedom from entangling alliances is best fitted to give thorough control of this implement. Here possibly we glimpse a category of utility or practicality which underlies and includes both vocational and cultural values as special cases or details.

Similarly one of the practical applications of quadratic equations commonly given is that of determining the time it would take two pipes to fill a tank if some relation between the rates of the two pipes is given. Here again actual use of this problem in practical life is not likely, but in the process of completing the square by the aid of which the problem is solved we again have the use of auxiliary quantity but in a more definite and controllable form than in geometry.

The broad, practical and cultural as distinguished from merely technical values which result from such a study of algebra may be illustrated thus. A certain business man, *A*, had a creditor, *B*, of doubtful solvency, who owed him $100. After much effort *A* succeeded in inducing *B* to give him a check for the $100 drawn on a certain bank. *A* presented the check at the bank, but found that *B*'s balance there was only $90. *A*, who had studied the subject of completing the square in algebra, promptly deposited $10 to the credit of *B* and had his check cashed, saving $90 by the operation. Every person has need to use algebra or algebraic methods, more or less similar to this every day.

In algebra we find other illustrations of the definite and progressive ways in which we can learn to use auxiliary quantities; in certain cases of factoring, in transposition (a disguised form of the use of auxiliary quantity), in the addition and subtraction method of elimination, in rationalizing denominators, in the use of variables with respect to limits, in the proofs or formulas for A.P. and G.P., in logarithms, etc.

So much, then, for some simple illustrations of the aspect of the subject which we are trying to present. We may now say a word as to the relation of this method to the other tendencies in the teaching of algebra, noticeable at the present time, which were named at the outset.

The treatment of algebra in the way just described it is believed does not conflict with the movement which would make the content of algebra more directly practical. Rather it welcomes every algebraic problem taken from the mechanic arts as a concentrated illustration and justification of itself. Reciprocally, appreciation of functional values gives a broader meaning and value to such concrete applications. Also the functional treatment should welcome every new refinement in the logic of the subject, since every such refinement rightly used is the source of new power.

However in this combination and coördination of methods some conflicts may arise and some adustments be called for. Thus a particular example or topic may be useful functionally but not be practical in the usual narrow sense of that term. For instance, it may be shown from a functional point of view that the identity sign is not needed or desirable in the early

study of algebra and this the logicist may not be willing to admit. However, this is a field which I do not desire to enter at this time.

In closing allow me to state the leading advantages in the functional or broad utility method of studying algebra, enlarging somewhat on those which have been mentioned incidentally in the preceding statement.

1. We have the removal of certain logical difficulties and extended power to deal with logical processes. Besides getting rid of difficulties the pupil is set free and gains power to leap from leading point to leading point and to fill in with exact equivalence relations later on.

Mr. H. V. Gummere, who is teacher of mathematics in Drexel Institute, Philadelphia, tells me that he has an evening class of the more mature mechanics from the Baldwin Locomotive Works and similar establishments. In one course of study given them, he has two hours a week for six months in which to teach the elements of analytical geometry and of differential and integral calculus, the aim being that after a man has taken the course he shall be able to continue his studies of the calculus alone and become able to read a stiff mathematical book on electricity. Mr. Gummere says that many of the men after taking the course are able to do this. The practical instinct in the men is so highly developed that they can quickly take in the broad outlines of the calculus in its functional aspect and go on and develop these themselves.

2. It is believed that the functional method increases the interest of pupils in vital ways and gives the subject something like a constant sense of worthwhileness. Often general utility considerations like those presented above seem to touch the average pupil at a more vital point and in a more stimulating way than direct concrete applications. The subject of interest needs a thorough working-out. Pupils are more interested in geometrical drawing than they are in going out on the campus and determining the height of a steeple or the distance of an inaccessible object; often more interested in working out puzzles than in either. The law seems to be that pupils are most interested in that form of activity which gives or promises to give largest returns for least effort. This may explain why general utility considerations appeal to them strongly.

3. It gives a vital and essential but not a costly concrete correlation of studies. This is a correlation of and through functions and utilities.

4. It is readily seen that all functions vary with circumstances, hence this treatment tears apart the narrow rigid processes of the demonstrational treatment, and leaves the pupil free and stimulates him to variations and innovations. This, rather than the demonstrational one, seems to be the route along which some of the greatest discoveries in mathematics have been made. Hence our method gives new value to the study of the history of a subject.

5. Finally the method under consideration helps to define the cultural values of education, and put these values in a more definite and controllable shape. This is a help which is very much needed at this time.

LAWRENCEVILLE SCHOOL,
LAWRENCEVILLE, N. J.

MODERN TENDENCIES IN THE TEACHING OF ALGEBRA.*

BY N. J. LENNES.

One of the most obvious facts about mathematics in our secondary schools is a very general dissatisfaction which is expressed on all sides. There is an alarming number of failures, especially in the first year of the high school, which argues that the pupils do not find the subject suited to their tastes and capacities. Instructors in the colleges and universities rarely miss an opportunity for declaring that their students came poorly prepared. The programs of teachers' meetings and the tables of contents of pedagogical journals are teeming with titles which assume that something is wrong.

Is this general faultfinding simply the natural condition of normal progress, the sign of healthy life, or is there at present an abnormal divergence between our practices and our best principles? and if the latter, what are the causes of this divergence? The main facts of the general history of pedagogy

*Read at the meeting of the New York Section.

during the last few decades and of the teaching of secondary mathematics during this time furnish the answers to these questions.

One of the chief characteristics of modern pedagogy is the importance which it places upon the study of the child itself,—its needs, interests and capacities. It has become a commonplace that in growing from childhood to maturity the individual passes through a series of stages, and that in each stage the intellectual as well as the physical powers are quite different from those of the stages that precede and follow. The problem that the modern educator has set himself is to make such demands upon the pupil as will result in the most complete and most spontaneous self-expression at each stage.

The spontaneity of this self-expression has come to be of the greatest moment. In days not long gone by, the course of study was prepared without much question as to whether the work demanded was of natural interest to the pupil. By stern command he was bid to do his task. The more distasteful the work the greater its disciplinary value. All this has been radically changed. We now know that a person, young or old, may be compelled to perform a certain amount of labor, physical or mental, but that so long as his interest is not awakened so long does he fall far short of doing the best work of which he is capable. President Eliot is often quoted as saying that any man who is worth his salt, whether he works with his brains alone or with his brains and hands, takes delight first in the work itself, second, in what he produces, and third, in what it will bring him. There is a stupendous difference in capacity and also in happiness between the man who works only at the bidding of another and the man whose work has taken possession of him.

In this respect all people, young and old, are alike. I still remember my introduction to quadratic equations. The teacher gave us the dimensions of an ordinary school slate and the area of its frame and required us to find the width of the frame. The next day we had all failed to solve this problem. We were convinced that our equipment in algebraic processes were insufficient and were quite put out that such a simple problem should be beyond us. Learning to solve a quadratic equation was a mere detail after this interest had been created. What would

have been the comparative effect if the teacher had given us a quadratic equation and asked us to solve it, possibly seeking to interest us by saying that we should wish to obtain a complete knowledge of algebra?

But if it be admitted that it is necessary to lay hold on the natural interests of the child in order to secure the best results then the question: What manner of creature is this child? becomes a momentous one. It is in attempting to answer this question that the modern investigators in pedagogy have been led to make such profound study of child life. The results of this study so far as they effect the questions here at issue are unequivocal and accord entirely with common sense.

What are the characteristics of the child of fourteen for whom we are to build our course in algebra? He is a bundle of activities, not a philosopher. He is interested in phenomena rather than in their explanation. Most of all is he interested in that which enables him to do things,—to express himself in activity. Most remote from his interests are theoretical considerations which to him are not connected with the teeming world in which he lives and which at every moment is flooding his receptive and curious mind with sensations.

The understanding of these simple facts has created a pedagogical awakening which has wrought fundamental changes in the teaching of nearly all subjects in the secondary schools. A few decades ago the beginner's Latin grammar was a complete treatise of that great language. The same text was used by the child studying his first lesson in Latin and the graduate student in the universities. Now the beginner's Latin book is decidedly juvenile to the mature student. There are frequent comparisons with the mother tongue, and even more constantly than is apparent from the texts does the good teacher maintain interest by making connection with facts of language which the child already knows. Thus the study of Latin is made less logical, but it is vastly more interesting, and it is also more profound in that it takes cognizance of the interrelations of languages and of general principles of language as distinct from the principles of one particular language.

The teaching of other subjects has undergone similar changes. English was a study of grammatical forms, of styles of speech and writing. A vast amount of time was spent in analyzing

sentences, classifying figures of speech and in memorizing the accepted canons of rhetoric. Here also the aim was to obtain a logically complete view. But little emphasis was laid on actual writing.

History was a record of the births and deaths of kings and queens—a chronicle of wars and of great battles. Now we attempt to memorize fewer dates, but seek to understand more clearly the forces that are at work. We try to comprehend the acts of men in distant times by observing men as they act now and by trying to place ourselves and others in the positions of those we are studying.

Physics consisted in the statement of a few fundamental facts and in the deduction from these of other facts. An accumulation of facts was the prime object. Now the laboratory has so changed the study of physics that the pupil of fifty years ago would not recognize it under the old name.

In all the subjects there has been a sacrifice in logical formality and in the number of facts that are lodged in the memory. At the same time there has been a vast gain in real insight. The subjects have been clothed with a human interest which was foreign to them and which is doing wonders for the spread of general intelligence. That the transition has brought some losses no one denies, but on the whole it is agreed that the net gain has been great. There is not the slightest sign of a movement to return to the old order of things. By far the greater part of the changes here enumerated have been brought about during the last twenty years. The description just given of the old order of things is largely taken from my own experience as a student in American high schools.

It is a remarkable fact that throughout this period of adjustment to a fundamentally new point of view the teaching of secondary mathematics has remained practically unchanged. This is particularly true of algebra, with which we are concerned this evening. To verify this statement one has but to compare the texts of today with those of twenty or even thirty years ago. There have been improvements in typography, in simplicity and perhaps in accuracy of statement and in the number and character of the exercises. There have been additions of new matter such as graphs and problems from physics. These are usually put into the books in the form of appendages, either as formal

appendices or tacked on at the end of chapters instead of being woven in as organic parts of the books. There have also been additions of new cases of factoring, new cases in simultaneous quadratics, etc. There is an evident effort to treat every possible case. Indeed one of the standard algebras of today reminds one of the Latin grammars or the rhetorics of twenty years ago.

The essential character of these books has remained unchanged. The topics considered, the order of their introduction and the general mode of treatment are precisely what they were a quarter of a century ago. Thus it comes about that our courses in algebra have descended to us from a time when the accepted principles of pedagogy were quite different from those we hold to-day. It is difficult to assign reasons why the teaching of secondary mathematics has adjusted itself more slowly to the requirements of modern pedagogy than that of any other great subject, but there can be little doubt that this exceptionally tardy conformity of practice to our best principles is the reason for the prevailing discontent noted at the beginning.

But not only has there been a general failure to keep step with the growing pedagogical principles; there has even been a movement in the opposite direction. The desire for completeness has ever been at work introducing new cases of abstract manipulation. This has affected particularly the first part of the course. The rivalry of authors to make their books "strong on factoring" has extended this subject beyond all reasonable limits for a beginner's course. Complicated fractions and equations containing fractions, long problems in multiplication and division have filled the early part of algebra with operations on abstract symbols so that now the first half of the first year's work in algebra contains practically no concrete problems. There are usually a few such problems at the very beginning, but these are so simple that they are solved before the study of algebra begins. From twelve to fifteen weeks, and often more, the child is made to study abstract theory and work formal exercises.

That this is not in conformity with our best principles of pedagogy is now pretty generally recognized. In no other subject is the pupil kept in such complete isolation from the rest of the world for so long a time. In beginning the study of languages, ancient or modern, our teachers

are constantly making connection with the mother tongue. In history and the study of government we are all the while making our departure from present conditions with which the child is acquainted. In the sciences the laboratory is as much a necessity as the teacher. Even in advanced courses in mathematics, given in the graduate schools of our great universities, the lecturer frequently shows the connection between the subject he is teaching and other subjects with which the students are acquainted independently of that particular course. I well remember attending a course of lectures on quaternions in which after a few days of purely theoretical work the professor showed how the theory he had developed could be used to make very elegant proofs of certain theorems in spherical trigonometry. The effect on this class of mature and comparatively expert mathematicians was unmistakable. The eyes shone brighter, the pens worked more rapidly. Suddenly their interest had been awakened. I believe it quite safe to say that nearly all of our well-known lecturers on higher mathematics seek constantly to make connection with the student's general information. Suppose that we were to attend a course in which the lecturer should begin by laying down his axioms and definitions and that then for fifteen weeks, four or five hours a week, he were to go on developing a symbolism peculiar to that course, proving theorems and working exercises in terms of that symbolism. Never once would he give us a problem which we could understand as a problem aside from the particular subject we were studying,—never once make connection with other subjects we know. I am quite certain that even with our mature years we should find it a great test of our endurance. Suffice it to say that the lecturer will not try the experiment. For one thing he is too good a teacher and for another he knows too well that his classes would soon dwindle to nothing. The people attending them have arrived at years of discretion. The children who attend our classes have nothing to say about such matters.

> "Theirs not to reason why,
> Theirs but to do and die,
> Into the valley of death
> Go the poor youngsters."

If it is admitted, and it is scarcely open to doubt, that the

child is interested rather in events than in their explanation,—that his interest is aroused by the dynamic rather than the philosophical aspect of things, then it follows that, in the beginning, algebra must be studied for what it can be made to do. The chief aim should be the solution of concrete problems and the study of such new operations as are needed in making these solutions rather than the construction of a theoretical doctrine as an end in itself. This does not mean that theoretical doctrines should be neglected or that drill exercises in symbolic operation should be avoided. But it does mean that new operations should be introduced only as needed in the solution of concrete problems. It is only as an algebraic operation enables a pupil to solve problems that he understands as problems independent of the algebra that it is of real interest to him. Healthy interest can never be maintained for any great length of time within the *logical* boundaries of an abstract subject like algebra. How true this is will appear to each of us as we recall our early experiences. While a boy from ten to fourteen I was made to think hard about geometrical symmetry because I wanted to make a sail-boat that should sail straight. Similar triangles were of the most absorbing interest because by means of them I was enabled to measure the distance to an inaccessible point on the opposite side of a lake.

Testimony to this same general effect is at hand on all sides. May I relate just an incident. My first year of teaching was in a country school. We had a class in algebra and the pupils were drilled in the good old fashion on the fundamental operations. Four years later I visited the community and in talking with one of the pupils of that class, then grown six feet tall, we recalled old times in the school-house and he remarked that he remembered just two things from that algebra. One was the problem of finding the time between conjunctions of Mars and Jupiter, the time of revolution of each being given. The other was the problem of finding how wide a strip a farmer must cut around a given field in order to cut a certain number of acres. He confessed that he had tried lately to solve the latter problem, but found he had to look up the method. Every operation in abstract symbols, except such as anyone can do almost the first time he sees them, had vanished from his mind though he was drilled most mercilessly in them, and you may rest assured that he would never look up a single one of them for its own sake.

This started me thinking about the character of our teaching perhaps more than any other one thing. The boy had appeared interested in the fundamental operations. He could factor anything in the book and took pride in puzzling out complicated examples which I unearthed from other books. To me it appeared that he was interested in the abstract operations quite as much as in the concrete problems. But what a tremendous difference between these two interests. The factoring was a game of solitaire with a little added interest because he could excel the others and gain my approval and perhaps their envy. But interest akin to the interest in a game of solitaire is not of the kind that we should seek to develop. It is not vital. It never lasts.

The width of the strip about the field was of natural and lasting interest. The event showed clearly enough that it laid a permanent hold on the mental anatomy of the boy while the others did not. What if we were to build our whole algebra around problems of this sort? Obviously there was no thought of financial gain or any so-called practical value associated with the solution of these problems. The boy was far too shrewd to dream that he might some day make a genteel living by solving problems of this sort. It was pure, healthy intellectual interest in the problem for its own sake. It is difficult to understand how anyone could ever suspect that such problems might be thought practical in the sense that there is a dollar dangling at the end of their solution. I, for one, certainly never had the faintest trace of a belief that in their industrial activity people would be likely to be called upon to solve just such problems as these. And even if the problems were practical in this sense, that would be no great point in their favor so far as creating interest in a healthy child is concerned. Boys and girls are not economically practical. The boy plays baseball with a will and never thinks of getting paid for it. I struggled for days to get a method for measuring the distance across a lake and surely I never thought of getting rich that way.

Our fundamental proposition is then that in the beginning we will as far as possible study only such algebraic processes as we find are needed in solving concrete problems. That has been the universal rule in arithmetic and we now propose to continue it into the algebra. This proposition decides at once

the character of the whole course, the inclusion and exclusion of subject matter, the order of the topics and the general mode of treatment. An interesting verification of this is that about three years ago to my knowledge at least three different groups of people worked out an outline for elementary algebra, and while there was no intercommunication on these particular points the results were in such close agreement as to make anyone not acquainted with the facts reasonably sure that they must have come from a common source. The fact was that they arose independently of each other but from common principles.

As a curious illustration of how prone we are to find a defense for what has become endeared to us by long association, be it ever so illogical, I remark here that in some quarters objections have been raised to the new order of topics necessitated by the new point of view. The chief reason given is that the new arrangement does not naturally call for as many drill exercises in mere manipulation as did the older arrangement. This is perfectly true. The fifteen or twenty weeks of beginning algebra with practically no applications—one continuous stretch at the very outset of from 100 to 150 pages of definitions, principles and abstract exercises—these must vanish if the principles here urged are accepted. But what about the arithmetics which are used in the last four years of the grades in this respect? They contain comparatively few purely drill exercises, and an overwhelming number of concrete problems. There does not seem to be any good reason why the student in beginning algebra should need such a surfeit of abstract formal exercises while in arithmetic he has no such need. There he got an excellently working concrete problem.

To see how the dominant idea of an author determines his mode of treatment let us consider the introduction of negative numbers. If the chief purpose is to present the subject in its logical aspect he comes upon the negative number by trying to solve such equations as $x + 2 = 0$, and obtains all the properties of the negative number directly as consequences of its definition. The author whose chief purpose it is to present the subject in its dynamic aspect finds concrete instances in which the negative number is useful. He then studies its properties in connection with these concrete things to which it is applied. In this re-

spect his study of the negative number is strictly analogous to the study of integers and fractions in the grammar school. Similar differences are observable in the treatment of other topics.

This mode of treatment naturally confines itself to a somewhat smaller body of algebraic operations. But the effect is to obtain complete practical mastery of these by using them in a very large number of different connections and by awakening a genuine interest in them. When different operations are crowded upon a child as rapidly as they must be in a course where they take "everything up to quadratics" in one year there is not sufficient time to digest each one. I can think of no better illustration of this than the usual treatment of radicals. We find from thirty to forty pages of theorems, discussion and abstract exercises giving a very elaborate treatment of all thinkable cases. There is not a single concrete application. Is it your experience that when the student comes to study geometry he can handle radicals with the facility which one might desire? My experience has been that he has no idea that the complicated symbols with which he worked in that overloaded course in algebra had anything whatever to do with the altitude of an equilateral triangle.

It is proposed that we study a much smaller body of theorems on radicals—just enough to solve quadratics and then use this small body of algebraic information in solving a large body of interesting problems. We shall not know quite so much but we shall surely know something. Radicals taught this way are much more likely to become a permanent part of the learner's equipment.

The endeavor to obtain encyclopedic knowledge the first time one studies a subject but too often leads to weakness and no solid acquisitions whatever. *Seek ye first a few important things and make sure that ye find them and all other things shall be added unto you.*

While the beginning of the study of algebra should emphasize its dynamic rather than its logical aspect, it is by no means intended that it should end there. When the child has become familiar with the main processes of algebra through their dynamic aspect, they are a part of his world as much as his information about Lincoln and Christopher Columbus. They

then become legitimate objects of study for their own sake. This brings us to the necessity of a general review of the whole subject. This review must obviously be from a more mature point of view than the first course. If the two courses are essentially from the same point of view, say from the same text, then it is evident that either the first course is too difficult or the review course is not sufficiently strong to demand the student's best efforts. When we reflect a bit it is perfectly obvious that the child could not take an interest in lengthy discussions on algebraic processes the first time he studies the subject. None of us, young or old, are in the habit of reflecting deeply on things of whose very existence we are hardly aware.

Recognizing fully the great gap between the child and the mature person, it is our purpose to bridge this gap so that the transit may be normal and healthy. Beginning with the dynamic elements of algebra, which is everywhere the child's element, we learn to use its processes in doing things that the child sees some sense in doing. As the processes become familiar they themselves become objects of interest and are then studied. The great desideratum is that a natural interest is maintained all along. In this way the pupil is stimulated to greater and more pleasurable efforts. His real acquisition is much greater though he does not know so many special cases the first year. His whole intellectual life is healthier in that he is keeping in constant touch with his own common sense.

We may summarize the whole program by saying that it consists in an effort to

HUMANIZE SECONDARY MATHEMATICS.

BROWN UNIVERSITY,
PROVIDENCE, R. I.

GENERAL SUGGESTIONS FOR MARKING PAPERS IN THE STATE EXAMINATIONS, BASED UPON A PASSING MARK OF SIXTY PER CENT.

By A. Latham Baker, Charles E. Bikle, Grace A. Bruce, Lao G. Simons, and Eugene R. Smith.
A Committee of the New York Section.

These figures are intended as general suggestions for the class of errors designated, an indication of the views of a number of markers of examination papers, to be modified by each marker according to the particular features of each answer paper.

Besides securing a greater degree of harmony among different markers, it is hoped to standardize the work of the individual marker, and harmonize the marking of one hour with that of the next.

The figures are based upon a total of 10 for the question; —4 indicating a deduction of 4; —4 +, of 4 or more; + 5 an allowance of 5, and so on. P.R.8 means apportioned on a basis of 8 for the question instead of 10.

The exponent attached to the rating indicates the number who have approved of that rating. Each reader can, if desired, strike an average for himself. It was thought that the components would be more instructive than the bare average.

These results have been submitted to the examination division at Albany and have received their general approval. Their divergence from the views expressed here is indicated in heavy type.

"With these slight exceptions [noted in heavy type] I would be willing to adopt your scheme as ours."

(Signed) Charles F. Wheelock,
Chief of Examination Division.

It is requested that every one interested send to the chairman of the committee, any additions, emendations, corrections or suggestions that may occur, with a view to a future report which can be based upon a very much larger number of suggestions. In fact let each person interested consider himself a member of the committee and send in his suggestions, both of

approval of the present figures, to increase their weight, or of suggested changes.

The committee makes no specific recommendations but lets the collected material make its own recommendations.

Interpretation of Terms.

1. Show how to = (in construction problems in geometry) give correct diagram with enough explanation (where needed) to show what is intended.

2. Show how to = (in algebra) give a reasonably clear outline of the method to be used, or numerical illustration; any reasonably clear explanation which will show that the student has a fairly clear idea of what to do.

3. Find = (in construction problems) same as 1.

4. Find = (in computation problems) get an answer with enough intervening work to show that the student has really worked for an answer.

5. Find = (in algebra) find at least *one* value of the unknown [5]: find all the answers that will fulfil the conditions [3].

6. Find a = find *one* (*e. g.*, mean proportional).

7. Simplify = condense to compact form with exponents united, integers in numerators and denominators; additions and subtractions completed. (No negative exponent in the answer.)

8. Simplify = (in radicals) bring out from under the radical sign all that will come out, and leaving thereunder no fraction, or fractional or negative exponent, the radical of the lowest order.

9. Explain = any reasonably clear explanation; complete logical sequence not demanded.

10. Solve = find all the values of the unknown.

11. Prove = (theorem in geometry) give a formal demonstration.

12. Prove = (fact in geometry, not one of regularly recognized theorems of the text-books) give a reasonably correct sequence of conclusions or equations leading to the result sought, but not necessarily a completely formal demonstration.

13. Prove = (equation in algebra) any correct sequence of equations which produces the given equation from a recognized one, or reduces the given equation to a recognizable one by a

series of reversable operations: formality in the deduction of consequences not demanded.

14. Interpret = any reasonably correct explanation which shows that the candidate has a fairly correct idea of the subject.

15. Define = same as interpret; critical exactitude in minutiæ or use of terms not demanded; but definition must be correct within a fairly broad use of terms conf. (III, B, 3, 4, III, C, 7, 8).

I. MISTAKES IN GENERAL METHOD. —5

A. *General.*

1. General method correct, but vitiated by sporadic errors of logic. -4^2, -5^4, -6^3
2. General method correct, but vitiated by errors of inconsequence, inconclusiveness, irrelevancy. -2^2, -3^2, -4^4
3. Misinterpretation of question, possible justification, though improbable. -3, -4^7
4. Misinterpretation of question, no justification. -5^2, -10^6

B. *Algebraic.*

1. Answer correct, but no intervening work, mental solution possible. -0^4, -2^2, -4^2, **—10**
2. Answer correct, but no intervening work, mental solution impossible. -8^2, -10^6, **—10**
3. Answer correct, but no intervening work, probable trial solution (*e. g.*, arithmetical mean). -4, -5^4, -6^2, -10, **—10**
4. Numerical example substituted for general proof. -6, -7, -8^5, -10, **—5**
5. Particular for general case. -5, -6, -7, -8^4, -10, **—5**
6. Arithmetical evolution of radicals instead of combination by similar radicals. -4^4, -5^4
7. Fundamental blunders, *e. g.*, $2y/y = y$. -2^3, -6^2, -7, -10
8. Fundamental blunders, $(x-a)/(x-a) = 0$. -3^3, -6^2, -7, -10
9. Fundamental blunders, unknown in the answer. -6^2, -7, -10^5
10. Failure to recognize mathematical absurdity of erroneous answer. -5^8

11. Wrong formula, subsequent algebraic work correct.
-5^6, -6, -7

12. Distribution of exponents over terms, *e. g.*,
$\sqrt{a^2-b^2}=a-b$; $(a+b)^2=a^2+b^2$.
-7^2, -10^6

13. Operating upon an algebraic expression as if it were an equation. -5^2, -6, $(-3$ to $10)^2$

C. *Geometry.*

1. Missing link in chain of proof.
$-2+^2$, -3, -4, -5^3, $-5+$

2. Correct proof of allied theorem, so closely allied as to dubitate intentional substitution.
-3^2, -4, -5^4, $-5+$

3. Using later theorem as authority when such order is possible though very improbable. -0^5, -0

4. Using later theorem as authority when the candidate's ability to use such order is very doubtful.
-0^2, -2^2, -3, -4, -0

5. Using later theorem as authority when such order is impossible. -5, -6^2, -10^5

6. Using special figure for general (suggestive of wrong or incomplete conception) thus altering the hypothesis. -5^6, $-5+^2$

7. Giving special diagram, though not using its special features in the argument (suggestive of hazy conception). -0^5, -2^3

8. Proving part of hypothesis in addition to otherwise correct proof of the conclusion.
-2^2, -3, -4^3, -5^2, -0

9. Incorrect diagram but bulk of demonstration correct (suggestive of memory work). -5^6, -6^2

10. Proof of converse theorem substituted. -5^6, -10^2, -5

11. Diagram unintelligible so that the text cannot be connected. -3, -5, -10^6

12. Imposing too many conditions on construction lines, proof vitiated. -6^2

13. Imposing too many conditions on construction lines, proof not vitiated (conditions merely redundant). -2^3, -3^4

14. Assuming consequences from the looks of the diagram. -3, -5, -8, -10^4
15. Asserting a previous consequence which has not been drawn. -2, -3, -5^5, $-5+$
16. Special instead of general case in a construction problem. -5^3, -8, -10^3, $-5+$

II. SECONDARY MISTAKES, ESPECIALLY WHERE THE GENERAL METHOD IS CORRECT, INCLUDING MISTAKES IN COPYING, FORM, USE OF SYMBOLS, ETC.

-1 to -3

A. *General.*

1. Question wrongly copied, correctly carried out, work not easier. -2, -3^6, -4
2. Failure to omit mathematically impossible solution (*e. g.*, extraneous roots). -1, -2^6, -3
3. General method correct, but vitiated by errors of transcription. -2^5, -3^3
4. Pairing wrong values in final statement of answer. -2^2
5. Signs of inequality reversed (*e. g.*, $>$ for $<$). -5^2

B. *Algebra.*

1. Method correct, but vitiated by initial error, easing the work somewhat. -5^6, -6, $-5+$
2. Method correct, but vitiated by initial error, not easing the work. -2^2, -3^5, -5
3. Interchange of requirements, solving for wrong element, algebra correct. -3^7, -5
4. Correct answer, vitiated by final blunder. -2^2, -3^2, $-3+$
5. Clerical error in transcribing from one line to another. -1^3, -2^5
6. Misuse of equality sign (*e. g.*, between corresponding expressions). -1^3, -2^5

C. *Geometry.*

1. Correct step but wrong authority quoted. -1^2, -5^2, $-2+$
2. Incorrect reference to diagram (*e. g.*, to "line AB" when the part evidently meant is not so lettered). -1^7, -2.
3. Giving special diagram for general (suggestive of wrong conception) though not using the special features. -0^5, -2^3

4. Citation of authority by number (meaningless, or guess-work, or bluff). -0, -1, -2
5. Analysis into hypothesis and conclusion incorrect, the following work correct (suggestive of memory work). -3^2, $-3+$, -4^3, -5^2
6. Auxiliary line wrongly made, with following text correct for correct diagram (suggestive of memory work). -2^3, -3^5
7. Citation of some original problem as authority. -0^2, $-0+$, -1^3, -2
8. Construction problem data misread, correctly carried out, no easier. -2, -3^5, -4, -5
9. Construction problem diagram incorrect (*e. g.*, chord drawn across the given arc instead of to the ends) but statement and proof correct for a correct diagram. -2^3, -3^5

III. Mistakes in Amount of Work; Work Incomplete, or Containing too much; Work Complete, but Some of the Steps of Like Value Incorrect.

Deduction to be made in proportion to amount omitted, minor omissions or insertions. -1 to -3

A. *General.*

1. Work correct as far as carried, not finished. P.R.8^2, $-2+$, P.R.^{3}
2. Work complete, some co-ordinate independent steps wrong. P.R.^{6}, P.R.8^2
3. Work complete, some sequential steps wrong, thus vitiating the answer. $+5$ for 1st, P.R.^{4}, P.R.8
4. Failure to recognize additional answers after 1st. -2^5, -3^3, -5
5. General method correct, but vitiated by omitted parts (suggestive of memory work). -4^2, -5^5, P.R.
6. Uncompleted work, homogeneous, all steps of equal value (*e. g.*, reduction, simplification). P.R.^{2}, P.R.^{6}
7. Uncompleted work, not homogeneous, the last step the important one. -5^7, -6
8. Uncompleted work, similar co-ordinate steps (*e. g.*, expansion of terms). $-2+^2$, P.R.^{4}, P.R.8^2
9. Uncompleted work, sequential steps (*e. g.*, addition

B. *Algebra.*

1. Answer correct, but left in unreduced form (*e. g.*, $\sqrt{1}$, $\sqrt{4}$, $3/\frac{3}{5}$). -1^8
2. First of several answers. $+7^6$, P.R.^{2}
3. Definition inadequate because of too general language (*e. g.*, a variable = a quantity which varies). -4, -5^7, -10
4. Definition inadequate, because of some latent essential lacking (*e. g.*, exponent = a figure above and to the right). -7, -8, -10^6, -10
5. Formula given in place of theorem called for (*e. g.*, binomial formula for the binomial theorem). -2^2, -3, -4^2, -5^2, -8
6. Initial statement of formulæ in a problem. $+2^7$, $+4$
7. Initial application of formula (*e. g.*, expansion by binomial theorem). $+3^2$, $+4^3$, $+5^3$
8. Initial application of formula; following steps (*e. g.*, simplification of terms in binomial theorem expansion). P.R.3, P.R.5, P.R.7^2
9. Intentional omission in transcribing (*e. g.*, dropping denominators). -5^8
10. Omission of $\pm$ in quadratics. -2^2, $-2+^2$, -3^3, -4
11. Failure to recognize the *existence* of more than one answer. -2^2, $-2+^2$, -3^3, -4
12. Initial statement of equations in algebraic problem. $+5$

C. *Geometry.*

1. Correct step in demonstration, but omission of authority. -1^2, -2^3
2. Authority suggested, but not formally stated (in a demonstration). -0^7, -1
3. Omission of hypothesis in part or in whole, but correct use of same. -1^7, -2
4. Redundancy in steps of proof. -0^5, -1^4, -2
5. Auxiliary line omitted in text, though made in diagram and there used. -1^5, -2, -3^2
6. Omission to draw final conclusion formally. -1^7, -2
7. Definition inadequate, because of too general language (*e. g.*, spherical angle = intersection of two arcs). -4, -5^7, -10

8. Definition inadequate because of some latent essential lacking (*e. g.*, slant height = distance between two base edges of the frustrum). -7, -8, -10^{6}, -10
9. Answer in computation examples correct, but no intervening work or diagram, or hint as to method. -5^{6}
10. Imposing too many conditions on construction line (auxiliary), though the unwarranted conditions not used and thus the proof not vitiated. -2^{3}, -3^{4}
11. Asserting previous consequence which has not been drawn though perhaps hinted at. -1^{2}, -2, -3^{3}, $-3+$
12. Deduction of consequences by too long a step, with correct authorities so far as given, but with lacunæ in the chain of argument. -1, -2^{2}, -3^{4}, -5
13. Introduction of irrelevant matter. -0, -1^{4}, $-1+^{2}$, -2
14. Introduction of unnecessary steps. -0^{2}, -1^{3}, $-1+^{3}$
15. Q.E.D. used before the final equation is interpreted into language as required by the theorem. -0, -1^{7}
16. Construction problem: diagram apparently correct, with construction more or less indicated, but with no explanation. -4^{5}
17. Construction problem: diagram correct, but required proof not given. -4^{2}, -5^{4}, $-4+$, -10
18. Computation problem, initial equations correct, subsequent algebra wrong. $+5$

IV. Mistakes in Calculation.

Important errors -5
Minor errors. -2

1. Balancing errors, right result (*e. g.*, mistakes in characteristics of logarithms). -5^{6}, -6^{2}
2. Computation problem in geometry, initial equations correct, subsequent algebra or arithmetic wrong. $+5$

THE TEACHING OF MATHEMATICS IN THE ELEMENTARY AND THE SECONDARY SCHOOL.*

By Arthur Sullivan Gale.

During recent years there has developed in this country a very real interest in the *teaching* of mathematics, as evidenced by the formation of numerous associations of teachers of mathematics. This interest may be traced to two principal sources. The great mathematical revival finding its expression in the rapid and healthy growth of the American Mathematical Society has had naturally a reactionary effect upon collegiate and then upon secondary instruction. This effect is seen in the effort to put upon a scientific basis the elementary parts of mathematics in their relation to the subject as a whole. At the same time, modern pedagogy holds as its principal thesis that both subject matter and manner of presentation must be arranged with reference to a psychological study of the pupil. Hence a further rearrangement of mathematical material is required with the express object of obtaining and retaining the student's interest. These two courses of the demand for improvement in mathematical instruction call for two lines of preparation on the part of the teacher, mathematical and pedagogical. A man may have studied a wide range of mathematical topics and yet have so poor a notion of how to present his ideas that it takes several years' experience to learn to teach; and in the meantime many of his pupils may discover, or believe that they have discovered, that they are so mentally deficient as to be unable to grasp mathematics. On the other hand, a man may learn something of the technique of teaching and be so ignorant of the principles of the science that his students do not obtain any idea of the spirit of mathematical studies. Such ignorance may be partially pardoned in the man who is forced to teach many different subjects; but it is, even at present, no novelty to find a teacher of mathematics only, who thinks that he requires his students to give a complete reason for every step in a geometrical demonstration. It is generally conceded that the normal schools have been unable to afford proper mathematical training for the high school teacher, and it is gratifying that the colleges are beginning to

* By J. W. A. Young. Longmans, Green and Co., 1907.

offer courses in mathematics arranged especially for those intending to enter the field of secondary instruction. Such courses should touch on many topics having an immediate bearing on elementary algebra and geometry and not ordinarily included in the courses usually offered to undergraduates, as well as some discussion of pedagogical principles. The proper person to conduct them has been aptly described by a well-known mathematician as "a mathematician sufficiently interested in his subject to publish occasional investigations, and who has the *pedagogic instinct.*"

There is no lack of literature on the pedagogy of mathematics in various periodicals, but the number of texts is so limited that Professor Young's book should be welcomed by all teachers of secondary mathematics. As indicated in the preface, the historical aspect of the subject is not considered, nor are topics of subject matter taken up except in illustrating various forms of presentation. Yet the last five chapters (pp. 189–346) contain many valuable suggestions relating to subject matter.

Chapter I, the study of the pedagogy of mathematics, is chiefly introductory, the first important discussion being in Chapter II, the purpose and value of the study of mathematics (pp. 9–52). It is the best and most comprehensive treatment of the topic which has come to the attention of the reviewer. While the author gives due prominence to the practical value of mathematics, considering the importance of its facts and its relations to nature and to general culture, he places chief emphasis on mathematics as a mode of thought of the utmost importance. "Throughout all mathematics, from the first numbers lisped in the nursery to the aged mathematician's last sigh, the chief end of mathematics is thought, not routine,—*natural* thought, exercising the powers of the thinker in an unforced and interested manner . . ." (p. 204). Pierce's definition, "Mathematics is the science of necessary conclusions," is used and a number of examples of reasoning of daily occurrence are examined in detail. There follows a brief consideration of eleven other functions of mathematics, to which might be added its disciplinary value in requiring continuous application throughout long intervals of time. It is the occasional relaxing and consequent falling behind which causes many of the failures in mathematics.

In an analysis of the book it is natural to associate the next

five chapters, whose headings are: Methods and modes, The heuristic method, The individual mode, The Perry movement: the laboratory method, and Miscellaneous points of method and mode. In the first of these, method is defined as the manner in which the subject matter is arranged and developed, and mode as the manner in which it is presented to the pupils. The methods considered are the synthetic, analytic, deductive, inductive, Socratic, heuristic, and laboratory, with emphasis on the pedagogic importance of analysis and induction. The examination, recitation, lecture, genetic, heuristic, individual and laboratory modes are explained, and in the discussion of their relative values we read that "The test of any, of every mode, is whether or not it is, at the time when it is employed, the mode which enables the teacher to give to the class, to every pupil, the most of himself, of his knowledge, of his experience, of his guidance, of his enthusiasm, of his inspiration" (p. 67). This sentence shows, perhaps as well as any other, the broad-minded point of view of the author. Throughout these chapters he presents the disadvantages as well as the advantages of a particular mode or method, so that the reader, to derive benefit from his perusal of the book, is forced to think for himself, a process as important for the teacher as for the pupil. The chapter on the laboratory method contains many helpful suggestions which can be used, and are being widely used, even by those who do not regard their class rooms as laboratories.

One point, perhaps, deserves especial mention, namely, the advocacy of the "free use of motion" in geometrical proofs. The reviewer would go even further and plead for a more consistent use of geometric transformations in both elementary and higher geometry. With very little of the technique of transformations a great deal may be accomplished, and much use may profitably be made not only of displacements but also of simple symmetry and similitude transformations. Recent foreign texts have made much of symmetry, and the homothetic transformation, illustrated by the pantograph (there should be one in every class-room in geometry), might well be made the basis of the consideration of similar figures. The theory of transformations underlies the whole domain of geometry in so fundamental a manner that a student may be introduced to many of the most far-reaching theorems of geometry through a study of the es-

sential properties of a few comparatively simple transformations. But while a free use of motion is desirable, the example used in slightly different form in two places (pp. 110 and 275) is unfortunate. It is the Playfair-Thibaut* proof (?) that the sum of the angles of a triangle is two right angles. Are we displaying "absolute truth toward the pupil" if we ask him to accept a proof which was believed by its originators to have answered the question associated for centuries with Euclid's parallel postulate?

If the answer to this question depended only upon the relation of non-Euclidean geometry to elementary instruction it would not be worth discussing, but it involves no less than a consideration of the general principles upon which the course in geometry is to be arranged. Professor Young's ideas on this point are contained in the following excerpts:

"Concrete geometry, in one form or another, may be a constant part of the work in arithmetic from the earliest years. The transition to strict demonstration should not be abrupt, but gradual. Quite a little of informal, but real, demonstration may be done in the last year or two before the secondary school, so that when the pupil nominally begins the study of demonstrative geometry in the secondary school he has already not only a large stock of geometric names and facts, but the spirit of demonstration well awakened" (p. 263).

"When should more formal geometry be begun? Whenever the pupil feels the need of it. Use intuition freely and prove what does not seem evident. The pupil may accept intuition as proof; that is not bad. Many great mathematicians have done so with happy results for the science. As long as he accepts intuitionally statements which are true, *go on*—things are progressing nicely. When his intuition fathers a falsity, *prove* to him that it is false (by concrete example or the like). This will arouse him to a sharp criticism of what constitutes a proof, *your* proof, and will do more to teach him the nature of a proof than dozens of routine demonstrations learned by rote, or even passively understood" (p. 264).

"The best course to be taken seems to be to base the work

* Playfair, 1813 (cf. Halsted's translation of the "Science Absolute of Space," by Bolyai, p. 67) and Thibaut, 1818 (cf. Killing, "Einführung in die Grundlagen der Geometrie," Vol. I, p. 7).

on *large* bodies of hypotheses, assuming many things which might be proved, but to make inferences from these hypotheses in strict accord with the laws of thought (logic). This does not mean that everything should be built up by strict deductive reasoning on the basis of the premises assumed, nor that the really deductive reasoning used should always be expressed in the most formalistic manner, but that when professing to be strict, the reasoning should really *be* so, and that there should always be a clear line drawn between what is proved and what is assumed as sufficiently plausible without proof, or accepted and used on the assertion of others (*e. g.*, teacher) that it can be proved. Absolute truth towards the pupil is the first essential.

"Let the idea of building up a system of propositions rigorously deduced from a set of irreducible assumptions (axioms) be abandoned, but let the distinction between what constitutes a *proof* in mathematics and what does not be all the more emphasized" (p. 123).

"The recent researches on axioms explode, once for all, all hope of teaching the child a logically perfect geometry in which all the results are deduced from a set of irreducible first principles, and *with it goes the only justification for adhering longer to the semblance of such a rigorous deductive system*" (reviewer's italics, p. 199).

Most mathematicians will be in hearty sympathy with nearly all of this, but the italicized sentence seems to the reviewer to be a case of *non sequitor*. Can we not apply, as it were, the "method of successive approximations"? Let the first approximation in the grammar schools have the object of familiarizing the pupil with "geometric names and facts" and "the spirit of demonstration" if possible. A second approximation with the same object may be desirable in the high school. The third approximation, which may legitimately, from the historical point of view, be called demonstrative geometry, might be built upon a large body of dependent axioms, consistent of course, and complete except for considerations which seem trivial to the pupil, *e. g.*, of three points on a line one is between the other two. The pupil may thus obtain the important notion, far too important to be sacrificed, of a "logically perfect geometry," even while realizing, perhaps not until late in the course, that his own study is incomplete in detail. If solid geometry is taken

in college there is opportunity for a fourth approximation, while the final approximation would be a graduate course for the specialist in the foundations of geometry.

The Playfair-Thibaut proof mentioned above might well be used in introductory work, but in demonstrative geometry, the introduction of a new axiom, either implicitly or explicitly, to prove but a single theorem which may be based upon an axiom frequently used, seems inadvisable; especially when, as in the case in hand, the new axiom is dependent upon the other in such a hidden fashion that specialists have failed to perceive the dependence.

Three short chapters on Preparation of teachers: mathematical clubs, The material equipment, and The curriculum in mathematics, separate the consideration of general pedagogical questions from the discussion of particular subjects. In the first one notes with some surprise the absence of projective geometry from the minimum list of subjects with which the teacher should be familiar. The last of the three emphasizes the desirability of teaching algebra and geometry simultaneously, and considers the co-ordination of mathematics with physics, a point that may easily be carried too far.

Chapter XI contains a discussion of definitions and axioms, showing clearly the distinction between different classes of mathematical terms and various points of view of the axioms. Not all will agree with the author's conclusion "to *use* the axiom when needed, perhaps without formulation, simply with a tacit or open 'of course'" (p. 200).

The following three chapters are devoted to the teaching of arithmetic, geometry, and algebra, respectively. The first of these, covering fifty-four pages, the longest chapter in the book, is the last to which many mathematicians would turn, and yet it is one of the most interesting. It contains many valuable suggestions which apply as well to other branches than arithmetic, and it is worth reading if only for the delightful story of the boy who obtained assistance from his parents. The teacher of arithmetic should read with especial care the pages on the algebraic and geometric sides of arithmetic. "It seems desirable that the study of geometric form be carried throughout the work in arithmetic with increasing thoroughness as the years pass, that the elements of literal arithmetic be introduced in such

measure as they can be utilized, the whole to be one coherent subject,—*arithmetic*" (p. 248).

(*To be continued.*)

NOTES AND NEWS.

THOSE MEMBERS OF THE ASSOCIATION who have not paid their dues up to and including this year which ends November 30, 1909, need not expect to receive further copies of THE MATHEMATICS TEACHER. Let every member who has not already done so send their dues at once to the Treasurer, Miss Emma H. Carroll, Philadelphia High School for Girls, Philadelphia, Pa.

REPORT OF THE TWELFTH MEETING OF THE ASSOCIATION AT SYRACUSE UNIVERSITY, APRIL 10, 1909.—*Committees.*—Reports of progress from the Committee on Algebra Syllabus and the Committee on Mathematics in Continuation Schools were read and approved and the committees were continued.

The Committee on Systematic Association Work was discharged.

Amendments.—Paragraph I. of Section II. was amended to read: "The Annual meeting shall be held at a time and place to be selected by the council."

It was announced that an amendment giving certain financial power to the editorial board of THE MATHEMATICS TEACHER, and providing for the term of office of the editors, would be voted on at the fall meeting.

Next Meeting.—The next meeting will be held at the College of the City of New York, at a date not yet determined.

The following new members were elected April 10, 1909:

S. BEAMAN, VERA H., A.B.; Gouverneur, N. Y.
P. BOWER, MARY ISABEL, Sc.B.; High School, Ridley Park, Pa.
S. BROWN, CLAUDE N., A.B.; Principal of High School, Pulaski, N. Y.
S. BROWN, JENNIE MAY, B.S.; High School, Bradford, Pa., 194 Jackson Ave.
S. BROWN, SUSIE D., M.Ph.; North Side High School, Syracuse, 1500 Park St.
S. COLEMAN, NELSON L., A.B.; High School, Binghamton, N. Y., 106 Prospect St.
S. ELDRED, ARVIE, A.B.; High School, Troy, N. Y.
S. GEORGIA, ANNA C., A.B.; Ellicottville High School, Sidney, N. Y.

S. LINDSEY, ARCHIBALD, A.B.; High School, Brushton, N. Y.
S. MILLS, DANIEL B., B.S.; High School, Canisteo, N. Y.
S. MONTGOMERY, EARL L., A.B.; High School, Adams, N. Y.
S. ROBERTS, RUTH PENELOPE, Ph.B.; Port Byron, N. Y.
S. ROLLS, NINA OLIVE, Ph.B.; High School, Carbondale, Pa., 8 Archbald St.
S. RUSHMER, CLARENCE E.; A.B.; Waterloo, N. Y.
S. VAN CAMPEN, MERRITT S., Ph.B.; Principal of High School, Damascus, Pa.
R. WATKEYS CHARLES W., A.M.; University of Rochester, 5 Amherst St.
R. WOODWARD, F. A., A.B.; High School, Fairport, N. Y.

Program.—The following papers were read and discussed: "A Few Algebra Methods," by Philip R. Dean, Curtis High School, New York; "The Effort to make Algebra Yield Fruit," by William A. Cornish, Cortland Normal School; "The Locus Problem," by Arthur Sullivan Gale, University of Rochester; "Intuition and Logic in Geometry," by William Betz, East High School, Rochester, N. Y.

EUGENE R. SMITH,
Secretary.

THE SPRING MEETING of the Philadelphia Section was held May 5, at the Central High School. The topic for discussion was "The Original in Geometry." Dr. Fletcher Darell, of Lawrenceville School, contributed a paper analyzing the methods of attack commonly used. Professor George Alvin Snook spoke on class-room methods which he found effective in teaching originals. Messrs. Miller, Moyer and Whitaker also discussed the subject.

The election of officers for the coming year resulted: *President,* G. Alvin Snook, Central High School; *Vice-President,* Edward Fitch, DeLancey School; *Secretary,* Elizabeth B. Albrecht, High School for Girls; *Executive Committee,* George H. Hallett, University of Pennsylvania; Harry Rothermel, Southern High School.

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